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The Additive log-Inverse Weibull Distribution: Properties and Applications

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ABSTRACT
Through this article, we investigate certain properties of an additive version of the log-inverse Weibull distribution including expressions for the cumulative distribution function, reliability function, hazard rate function, quantile function, raw moments, incomplete moments etc. Some structural properties of the distribution are considered along with the distribution and moments of its order statistics. The maximum likelihood estimation of its parameters and the elements of the Fisher information matrix are obtained. Further, the effciency of the distribution as a distributional model is illustrated using two real life datasets. Moreover, the asymptotic behaviour of the maximum likelihood estimators are examined with the help of simulated datasets.

KEYWORDS
Maximum likelihood estimation, Model selection, Moments, Order Statistics, Simulation

1. Introduction

The inverse Weibull distribution (IWD), introduced by [12] through the cumulative distribution function ($c.d.f.$)

$$Q_1(x) = exp(-x^{-c}), \tag{1}$$

for any $x > 0$ and $c > 0$ and its related versions have been frequently used for modelling survival datasets in the areas of medicine, reliability, ecology, industry etc. Various modifications of the IWD and their applications have been investigated in literature by several authors like [6], [20], [13], [14], [10], [5], [23], [4], [8], and [16]. The additive Weibull distribution ($AWD(\rho, \sigma; \alpha, \beta)$) having the c.d.f.

$$Q_2(x) = 1 - exp\Big[-(\rho x^{\alpha} + \sigma x^{\beta})\Big], \tag{2}$$

for any $x \in \Re^+ = (0, \infty)$ with scale parameters $\rho > 0$, $\sigma > 0$ and location prameters $\alpha > 0$, $\beta > 0$ (such that $\alpha \neq \beta$) was proposed by [26] by combining the survival func-

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tions of two Weibull distributions. The distribution was later studied in detail by [19]. Moreover, truncated versions of various distributions, including the Normal distribution, Weibull distribution, Lindley distribution etc. have found wide applications in various areas of survival analysis and reliability theory. For example see [3], [2], [27], [24], [15],etc. Recently, a log-transformed version of the IWD having the name "the log-inverse Weibull distribution (LIWD)" along with its location-scale extended form, "the extended log-inverse Weibull distribution (ELIWD)", capable for modelling truncated datasets were investigated by [17] through their c.d.f.s

$$Q_3(x) = exp\{-[ln(x)]^{-c}\} \tag{3}$$

and

$$Q_4(y) = exp\left\{-b^{-1}[ln(x) - a]^{-c}\right\}, \tag{4}$$

respectively for $x \geq 0$ and $y \geq e^a$ with $a \in (-\infty, \infty)$, $b > 0$ and $c > 0$ respectively.

Through this paper, we consider an additive form of the ELIWD using the name "additive log-inverse Weibull distribution (ALIWD)" by combining the survival functions of two 2-parametric versions of the ELIWD, thereby increasing the flexibility of the model in handling survival data sets arising from a variety of fields including medicine, finance, geology, demography and engineering sciences. We try to establish that the ALIWD possesses more variability in terms of measures of central tendencies as well as skewness and kurtosis and has a number of shapes for its hazard rate function.

The paper is organized as follows: The definition of the ALIWD along with some of its important properties are presented in Section 2. In Section 3, we discuss some structural properties of the distribution while Section 4 presents the distribution and moments of its order statistics. Section 5 deals with the maximum likelihood (ML) estimation of the parameters of the ALIWD along with the derivation of its Fisher information matrix. In section 6 certain applications of the distribution to real life datasets are presented and the asymptotic behaviour of the ALIWD is examined with the help of simulated datasets in Section 7.

2. Definition and Properties of the Additive Log-Inverse Weibull Distribution

The definition and some important properties of the additive log-inverse Weibull distribution are provided in this section.

Definition 2.1. A continuous random variable U is said to follow the additive log-inverse Weibull distribution with parameters $\rho > 0, \sigma > 0, \alpha > 0$ and $\beta > 0$ if its c.d.f. $F_U(u)$ is of the following form, for $u > 1$.

$$F_U(u) = \exp\left\{-\left\{\rho[\ln(u)]^{-\alpha} + \sigma[\ln(u)]^{-\beta}\right\}\right\} \tag{5}$$

A distribution with c.d.f. (5) is henceforth represented as the ALIWD(ρ, σ, α, β) having shape parameters α, β and scale parameters ρ, σ. In order to ensure the identifiability of the ALIWD(ρ, σ, α, β), we assume $\alpha \neq \beta$ whenever necessary. Clearly, depending on whether $\rho = 0$ or $\sigma = 0$, the ALIWD(ρ, σ, α, β) reduces to the ELIWD(0, $\sigma^{-c^{-1}}$, β) or ELIWD(0, $\rho^{-c^{-1}}$, α) respectively.

The expressions for the probability density function (p.d.f.), survival function, hazard rate function and the reverse hazard rate function of the ALIWD are obtained through the following theorem, the proof of which follows directly from (5).

Theorem 2.2. *We obtain the p.d.f. $f_U(u)$, the survival function $\overline{F}_U(u)$, the hazard rate function $h_U(u)$ and the reverse hazard rate function, $\tau_U(u)$ of the ALIWD(ρ, σ, α, β) as given below for $u > 1$.*

$$f_U(u) = u^{-1}\left(\alpha\rho(\ln(u))^{-\alpha-1} + \beta\sigma(\ln(u))^{-\beta-1}\right)\exp\left[-\left\{\rho[\ln(u)]^{-\alpha} + \sigma[\ln(u)]^{-\beta}\right\}\right], \tag{6}$$

$$\overline{F}_U(u) = 1 - \exp\left[-\left\{\rho[\ln(u)]^{-\alpha} + \sigma[\ln(u)]^{-\beta}\right\}\right], \tag{7}$$

$$\begin{aligned} h_U(u) &= u^{-1}\left(\alpha\rho(\ln(u))^{-\alpha-1} + \beta\sigma(\ln(u))^{-\beta-1}\right)\exp\left[-\left\{\rho[\ln(u)]^{-\alpha} + \sigma[\ln(u)]^{-\beta}\right\}\right] \\ &\quad \times\left\{1 - \exp\left[-\left\{\rho[\ln(u)]^{-\alpha} + \sigma[\ln(u)]^{-\beta}\right\}\right]\right\}^{-1} \end{aligned} \tag{8}$$

and

$$\tau_U(u) = u^{-1}\left(\alpha\rho(\ln(u))^{-\alpha-1} + \beta\sigma(\ln(u))^{-\beta-1}\right). \tag{9}$$

The behaviour of the p.d.f., $f_U(u)$ when $u \longrightarrow 1$ and when $u \longrightarrow \infty$ can be summarised as

$$\lim_{u\to 1} f_U(u) = \begin{cases} \infty, & 0 < \alpha < \beta < 1 \\ 0, & \alpha < 1, \beta > 1 \\ 0, & 1 < \alpha < \beta < \infty \end{cases}, \tag{10}$$

and

$$\lim_{u\to\infty} f_U(u) = 0 \tag{11}$$

respectively for all values of ρ and σ.

Also, on differentiating (6) and (8) with respect to u, we have

$$f'_U(u) = f_U(u)u^{-1}\left\{\psi_1(u;\underline{\theta}) - \psi_2(u;\underline{\theta})\left[\psi_1(u;\underline{\theta})\right]^{-1} - 1\right\} \tag{12}$$

and

$$h'_U(u) = h_U(u)u^{-1}\left[\psi_1(u;\underline{\theta})\left[1 - exp{-}\psi(u;\underline{\theta})\right]^{-1} - \psi_2(u;\underline{\theta})\left[\psi_1(u;\underline{\theta})\right]^{-1} - 1\right], \tag{13}$$

in which $\psi(u;\underline{\theta}) = \left\{\rho[\ln(u)]^{-\alpha} + \sigma[\ln(u)]^{-\beta}\right\}$, $\psi_1(u;\underline{\theta}) = \left(\alpha\rho(\ln(u))^{-\alpha-1} + \beta\sigma(\ln(u))^{-\beta-1}\right)$ and $\psi_2(u;\underline{\theta}) = \left(\alpha(\alpha-1)\rho(\ln(u))^{-\alpha-2} + \beta(\beta-1)\sigma(\ln(u))^{-\beta-2}\right)$. Based on (12) and (13), we have the following remarks.

Remark 1. From (12), it can be observed that $f_U(u)$ is a non-decreasing (or non-increasing) function of u if $\psi_2(u;\underline{\theta})$ is greater than (or less than) $\psi_1(u;\underline{\theta})\left\{[\psi_1(u;\underline{\theta})] - 1\right\}$.

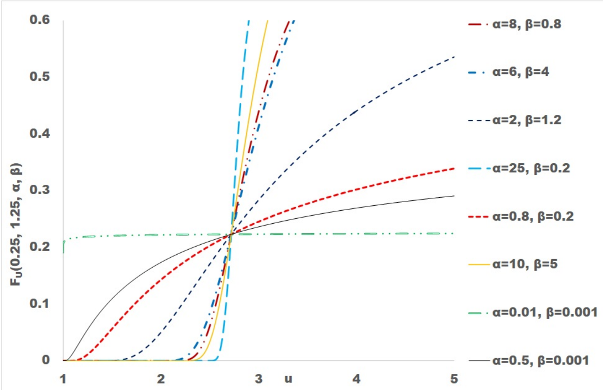
Remark 2. On simplifying (13), it can be observed that $h_U(u)$ is a non-increasing (or non-decreasing) function of u if $[\psi_1(u;\underline{\theta})]^2\left\{\psi_1(u;\underline{\theta}) + \psi_2(u;\underline{\theta})\right\}^{-1}$ is less than (or greater than) $1 - \exp[\psi(u;\underline{\theta})]$.

Remark 3. The modes of the ALIWD(ρ, σ, α, β) are the solutions of the equation $f'_U(u) = 0$, which reduces to

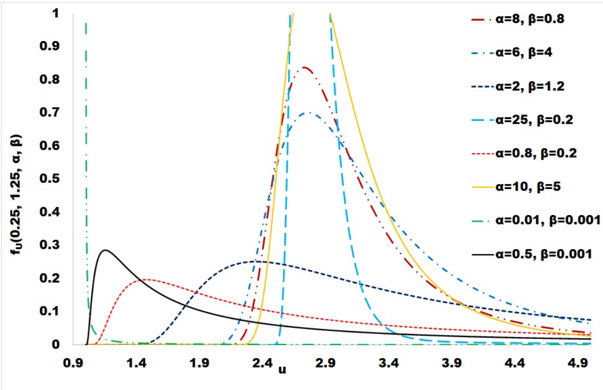
$$\psi_2(u;\underline{\theta})\left\{\psi_1(u;\underline{\theta})\right\}^{-1} = \psi_1(u;\underline{\theta}) - 1. \tag{14}$$

There may be more than one roots for (14). The root of (14) corresponds to a local max-

Figure 1. Plots of the c.d.f. and p.d.f. of the ALIWD(ρ, σ, α, β) for particular values of its parameters.



(a) Plots of c.d.f. for various values of α and β and ρ =0.25, σ = 1.25.



(b) Plots of p.d.f. for various values of α and β and ρ =0.25, σ = 1.25.

imum, a local minimum or a point of inflexion depending on whether $\frac{d^2 \ln f_U(u)}{du^2} < 0$, $\frac{d^2 \ln f_U(u)}{du^2} > 0$ and $\frac{d^2 \ln f_U(u)}{du^2} = 0$, in which

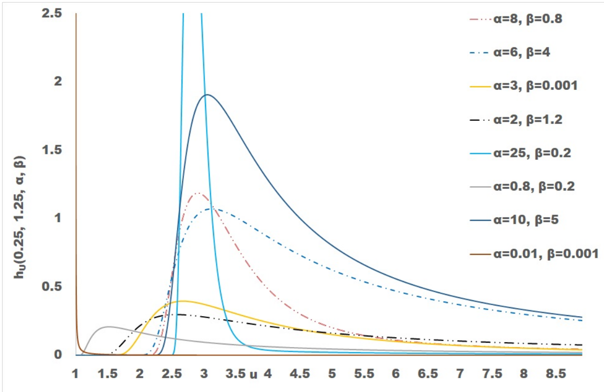
$$\frac{d^2 \ln f_U(u)}{du^2} = \frac{u^{-2}}{\psi_1(u;\underline{\theta})} \left\{ \left(\psi_2(u;\underline{\theta}) + \psi_1(u;\underline{\theta})\right)\left(1 - \psi_1(u;\underline{\theta})\right) - \psi_2(u;\underline{\theta})^{-2}\psi_1(u;\underline{\theta})^{-1} + \psi_3(u;\underline{\theta}) \right\},$$

with $\psi_3(u;\underline{\theta}) = \left(\alpha(\alpha+1)(\alpha+2)\rho(\ln(u))^{-\alpha-3} + \beta(\beta+1)(\beta+2)\sigma(\ln(u))^{-\beta-3}\right)$.

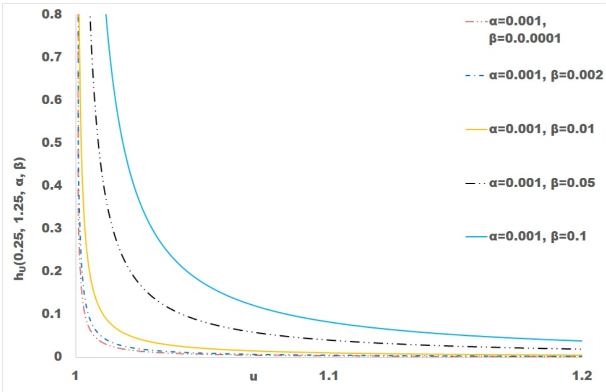
We have presented the plots of the c.d.f. and the p.d.f. of the ALIWD(ρ, σ, α, β) for choices of its parameters in Figure (1) and those of its hazard rate function in Figure (2) From the plots, we can observe the following aspects with regards to the characteristics of the distribution.

- The point of intersection of the cdf os the ALIWD(ρ, σ, α, β) is $(2.718282, \exp[-(\rho+\sigma)])$, which indicates that there is a probability of $[1 - \exp[-(\rho+\sigma)]]$ that an ALIWD(ρ, σ, α, β) distributed life time is at least 2.7182, for any values of the parameters α and β.
- The p.d.f. of the ALIWD(ρ, σ, α, β) is a non-increasing function of u for extremely small values of its parameters α and β. As the values of α and β increase, the distribution

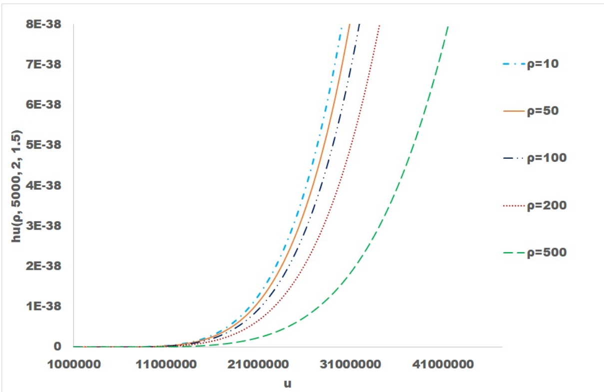
Figure 2. Plots of the hazard rate function of the ALIWD(ρ, σ, α, β) for particular values of its parameters.



(a) Plots of $h_U(u)$ for various values of α and β when ρ =0.25, σ = 1.25.



(b) Plots of $h_U(u)$ for extremely small values of α and β when ρ =0.25, σ = 1.25.



(c) Plots of $h_U(u)$ for very large values of ρ or σ and small values of α and β.

becomes uni-modal and the distribution tends to be more symmetric.

- It can be observed that the hazard rate function of the ALIWD(ρ, σ, α, β) takes different shapes based on the values α and β. When both α and β are extremely small, $h_U(u)$ is a decreasing function of u and it is a non-decreasing function of U for extremely large values of α and β. For all other values of the parameters, the hazard rate function has the upside-down bathtub shape.

Quantile Function:
The quantile function of the ALIWD(ρ, σ, α, β) is obtained by inverting $F_U(u) = \exp\left[-\left\{\rho[\ln(u)]^{-\alpha} + \sigma[\ln(u)]^{-\beta}\right\}\right] = p$, where $p \in (0, 1)$. We arrive at the non-linear equation

$$\rho[\ln(u)]^{-\alpha} + \sigma[\ln(u)]^{-\beta} = -\ln(p). \tag{15}$$

By using the substitution $t = [\ln(u)]^{-1}$, we have

$$\rho t^{\alpha} + \sigma t^{\beta} = -\ln(p) = y. \tag{16}$$

Expanding the term t^{α} using the Taylor series expansion, we have
$t^{\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}(t-1)^k = \sum_{j=0}^{\infty} \nu_{1j}t^j$, in which $\nu_{1j} = \sum_{k=j}^{\infty} \frac{(-1)^{k-j}}{k!}\binom{k}{j}\alpha^{[k]}$, $(\alpha)_k$ is the descending factorial and $(\alpha)^k$ denotes the ascending factorial.
Expanding t^{β} in similar terms with $\nu_{2j} = \sum_{k=j}^{\infty} \frac{(-1)^{k-j}}{k!}\binom{k}{j}(\beta)_k$ and substituting in (16), we have

$$y = S(t) = \sum_{j=0}^{\infty} \nu_j t^j, \tag{17}$$

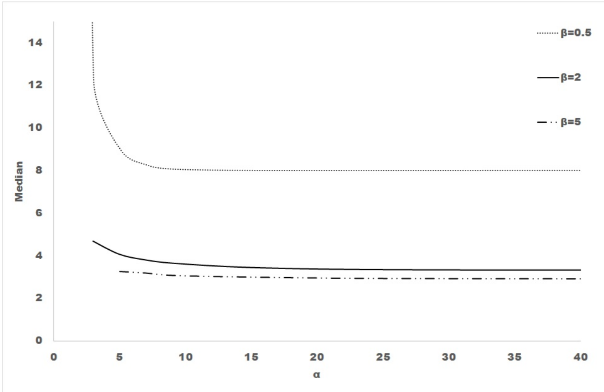
where $\nu_j = \rho\nu_{1j} + \sigma\nu_{2j}$.
Using the Lagrange theorem under the assumption that the power series expansion holds, we have
$y = S(t) = \nu_0 + \sum_{j=1}^{\infty} \nu_j t^j$, where $\nu_1 = S'(t) \neq 0$ and $S(t)$ is analytic at a zero point. Then, the inverse power series expansion, $t = S^{-1}(y)$ exists and is given by
$t = S^{-1}(y) = \sum_{j=1}^{\infty} \vartheta_j y^j$, in which $\vartheta_j = \frac{1}{n!}\left\{\frac{d^{j-1}}{dt^{j-1}}[\Psi(t)]^j\right\}_{t=0}$, $\Psi(t) = \frac{1}{S(t) - \nu_0}$.
Now the quantile function can be written as

$$Q(p) = \exp\left[\sum_{j=1}^{\infty} \vartheta_j[-\ln(p)]^j\right]^{-1}. \tag{18}$$

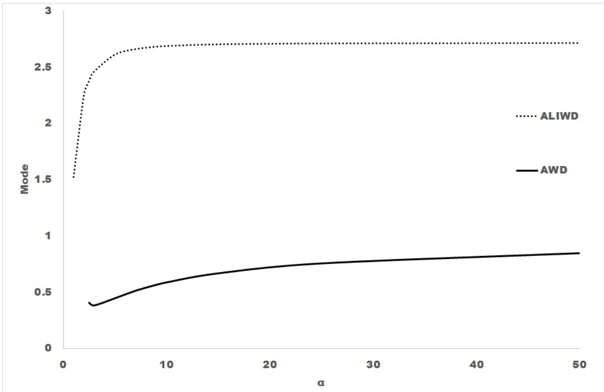
Using the expansion, $[-\ln(1-(1-p))]^j = j!\sum_{n=j}^{\infty} \frac{(-1)^{j+n}}{n!}S(n, j)(1-p)^n$, where $S(n, j)$ is the Stirling numbers of the first kind for $n = 1, 2, \ldots$, $j = 0, 1, \ldots, n$ and satisfying the recurrence relation $S(n+1, j) = S(n, j-1) - nS(n, j)$ in (18), we have the quantile function as

$$Q(p) = \exp\left[\sum_{j=1}^{\infty} q_j(1-p)^j\right]^{-1}, \tag{19}$$

Figure 3. Plots of the mode and the median of the ALIWD(ρ, σ, α, β) for particular values of its parameters.



(a) Plots of the mode of the ALIWD(1, 1, 0.5, β) for varying values of α and particular values of β.



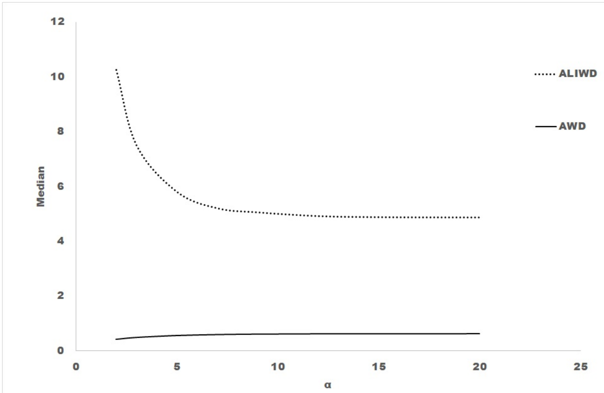
(b) Plots of the median of the ALIWD(1, 1, 0.5, β) for varying values of α and particular values of β

where $q_j = \dfrac{1}{n!} \sum\limits_{j=1}^{n} \vartheta_j j!(-1)^{j+n} S(n, j)$, for $n = 1, 2, \ldots$.

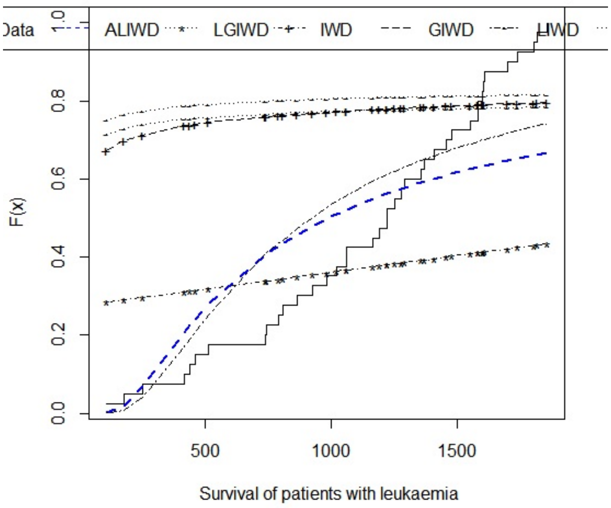
Clearly, the median of the ALIWD(ρ, σ, α, β) is obtained by using the substitution $p = 0.5$ in (15). We have plotted the median and mode of the ALIWD(ρ, σ, α, β) for varying values of its parameters and presented them in Figure (3). Moreover, for comparison purposes, the median and the mode of the ALIWD(ρ, σ, α, β) is plotted along with those of the AWD(ρ, σ, α, β) for fixed arbitrary values of α and β in Figure (4). From both the figures, it can be observed that the median of the ALIWD(ρ, σ, α, β) is a non-increasing function while that of the AWD(ρ, σ, α, β) is a non-decreasing function of α for fixed values of β. Also, the modes of the ALIWD(ρ, σ, α, β) show greater flexibility for varying values of α and fixed arbitrary values of β.

Raw moments:

Figure 4. Plots of the mode and median of the ALIWD(ρ, σ, α, β) and the AWD(ρ, σ, α, β) for particular values of their parameters.



(a) Plots of the mode of the ALIWD(1, 1, α, 0.5) and the AWD(1, 1, α, 0.5) for particular values of its parameters.



(b) Plots of the median of the ALIWD(1, 1, α, 0.5) and AWD(1, 1, α, 0.5) for particular values of its parameters.

By definition, the r^{th} raw moment $\mu_r^{'}$ of the ALIWD(ρ, σ, α, β) is

$$\mu_r^{'} = E(U^r) = \int_1^\infty u^r u^{-1}\left(\alpha\rho(\ln(u))^{-\alpha-1} + \beta\sigma(\ln(u))^{-\beta-1}\right) \tag{20}$$

$$\exp\left[-\left\{\rho[\ln(u)]^{-\alpha} + \sigma[\ln(u)]^{-\beta}\right\}\right]du. \tag{21}$$

Using the substitution $y = [\ln(u)]^{-1}$ in (20),

$$\mu_r^{'} = \int_0^\infty exp\left[ry^{-1}\right]\left(\alpha\rho y^{\alpha-1} + \beta\sigma y^{\beta-1}\right)\exp\left[-\left(\rho y^{\alpha} + \sigma y^{\beta}\right)\right]dy \tag{22}$$

$$= \sum_{k=0}^{\infty}\frac{r^k}{k!}\int_0^\infty y^{-k}\left(\alpha\rho y^{\alpha-1} + \beta\sigma y^{\beta-1}\right)\exp\left[-\left(\rho y^{\alpha} + \sigma y^{\beta}\right)\right]dy.$$

Substituting $y^{\alpha} = t$, we have

$$\mu_r^{'} = \alpha^{-1}\sum_{k=0}^{\infty}\frac{r^k}{k!}\int_0^\infty t^{-k\alpha^{-1}}\left(\alpha\rho t^{(\alpha-1)\alpha^{-1}} + \beta\sigma t^{(\beta-1)\alpha^{-1}}\right)\exp\left[-\left(\rho t + \sigma t^{\beta\alpha^{-1}}\right)\right]dt$$

$$= \alpha^{-1}\sum_{k=0}^{\infty}\frac{r^k}{k!}\left\{\alpha\rho J_{\left(-k\alpha^{-1};\rho,\sigma,\beta\alpha^{-1}\right)} + \beta\sigma\alpha^{-1}J_{\left((\beta-k)\alpha^{-1}-1;\ \rho,\ \sigma,\ \beta\alpha^{-1}\right)}\right\}, \tag{23}$$

in which

$$J_{(s;\ a,\ b,\ c)} = \int_0^\infty u^s \exp -(au + bu^c)du. \tag{24}$$

As illustrated by [19], (24) can be evaluated in different ways depending on the values of c, in order to find explicit expressions for $\mu_r^{'}$.
When $c = \dfrac{\beta}{\alpha}$, where $\beta \geq 1$, $\alpha \geq 1$ and $\alpha \neq \beta$ are relatively prime natural numbers and when $0 < c < 1$ (*i.e.* $\beta < \alpha$) and $c > 1$ (*i.e.* $\beta > \alpha$), we use Eq. (2.3.2.13) of Vol.1 of [21](p 321).
When $\beta < \alpha$ we have

$$J_{(s;a,b,c)} = \sum_{m=0}^{\alpha-1}\frac{(-b)^m\Gamma(s+1+m\beta\alpha^{-1})}{m!a^{\left(s+1+m\beta\alpha^{-1}\right)}}$$
$$\times\ {}_{(\beta+1)}F_{\alpha}\left(1, \Delta(\beta, s+1+m\beta\alpha^{-1}); \Delta(\alpha, 1+m); \frac{(-1)^{\alpha}\beta^{\beta}b^{\alpha}}{\alpha^{\alpha}a^{\beta}}\right), \tag{25}$$

for $s + 1 + \dfrac{m\beta}{\alpha} > 0$ in which the generalised hypergeometric function

${}_dF_b(p_1, p_2, ...p_\beta; q_1, q_2, ...q_\alpha; u)$ and $\Delta(p, q)$ are defined as

$$ {}_\beta F_\alpha(p_1, p_2, ...p_\beta; q_1, q_2, ...q_\alpha; x) = \sum_{k=0}^{\infty} \frac{(p_1)_k(p_2)_k...(p_\beta)_k x^k}{(q_1)_k(q_2)_k...(q_\alpha)_k k!} \tag{26} $$

and

$$ \Delta(p, q) = (q/p, (q + 1)/p, ..., (q + p - 1)/p). \tag{27} $$

When $\beta > \alpha$ and $(s + 1 + m)\alpha > 0$ we have

$$ \begin{aligned} J_{(s;a,b,c)} = &\sum_{m=0}^{\beta-1} \frac{(-1)^m \alpha\Gamma((s + 1 + m)\alpha\beta^{-1})}{\beta m! b^{((s+1+m)\alpha\beta^{-1})}} \\ &\times {}_{(\alpha+1)}F_\beta\left(1, \Delta(\alpha, (s + 1 + m)\alpha\beta^{-1}; \Delta(\beta, 1 + m); \frac{(-1)^\beta \alpha^\alpha a^\beta}{\beta^\beta b^\alpha}\right). \end{aligned} \tag{28} $$

For irrational c, an approximation of vanishingly small error can be made in (25) and (28), using increasingly rational approximation for this parameter. When $\beta \geq 1$ and $\alpha \geq 1$ are co-prime natural numbers we can evaluate (24) using the Meijer $G_{d,b}^{m,n}$ function as defined below.

$$ G_{\beta,\alpha}^{m,n}\left(z \,\middle|\, \begin{matrix} p_1, \ldots, p_\beta \\ q_1, \ldots, q_\alpha \end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(q_j + t) \prod_{j=1}^n \Gamma(1 - p_j - t)}{\prod_{j=n+1}^\beta \Gamma(p_j + t) \prod_{j=m+1}^\alpha \Gamma(1 - q_j - t)} z^{-1} dt, \tag{29} $$

where $i = \sqrt{-1}$ and L represents an integration path as in [9]. Also,

$$ \exp(-bx^{\beta/\alpha}) = G_{0,1}^{1,0}\left(bx^{\beta/\alpha} \,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right). \tag{30} $$

Then,

$$ J_{(s;a,b,c)} = \frac{\beta^{s+1/2}}{(2\pi)^{((\alpha+\beta)/2-1)} a^{s+1}} G_{\beta,\alpha}^{\alpha,\beta}\left(\frac{b^\alpha \beta^\beta}{a^\beta \alpha^\alpha} \,\middle|\, \begin{matrix} \frac{-s}{\beta}, \frac{(-s+1)}{\beta}, \ldots, \frac{(-s+\beta-1)}{\beta} \\ 0 \end{matrix}\right). \tag{31} $$

As a special case, when $\alpha = 1$, according to Eq.(9.31.2) in [9],

$$ G_{\beta,\alpha}^{m,n}\left(z^{-1} \,\middle|\, \begin{matrix} p_r \\ q_s \end{matrix}\right) = G_{\alpha,\beta}^{n,m}\left(z \,\middle|\, \begin{matrix} 1 - q_s \\ 1 - p_r \end{matrix}\right). \tag{32} $$

In the light of (24) caqn be written as

$$ J_{(s;a,b,c)} = \frac{\beta^{s+1/2}}{(2\pi)^{((\beta-1)/2)} a^{s+1}} G_{1,\beta}^{\beta,1}\left(\frac{a^\beta}{b\beta^\beta} \,\middle|\, \begin{matrix} 1 \\ \frac{(s+1)}{\beta}, \frac{(s+2)}{\beta}, \ldots, \frac{(s+\beta)}{\beta} \end{matrix}\right). \tag{33} $$

Generally, for any positive real numbers α and β such that $c > 1$ and $\frac{s+m+1}{c} > 0$, by expanding $exp(-au)$ in (24), we have

$$J_{(s;a,b,c)} = \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} \int_0^{\infty} u^{s+m} \exp\left[-(bu^c)\right] du$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{c\, m!} \left(\frac{a}{b^{1/c}}\right)^m \frac{1}{b^{(s+1)/c}} \Gamma\left(\frac{s+m+1}{c}\right). \quad (34)$$

Following [25], the Wright generalised hypergeometric function can be defined as

$${}_d\Psi_b\left[\begin{array}{l}(p_1,P_1),\ldots,(p_d,P_d)\\(q_1,Q_1),\ldots,(q_d,Q_b)\end{array};z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{d}\Gamma(p_j+P_jn)}{\prod_{j=1}^{b}\Gamma(q_j+Q_jn)} \frac{z^n}{n!}, \quad (35)$$

for $1+\sum\limits_{n=0}^{\infty} Qj - \sum\limits_{n=0}^{\infty} Pj > 0$. Using (35) in (34) we obtain

$$J_{(s;a,b,c)} = \frac{1}{c\, b^{(s+1)/c}}\, {}_1\Psi_0\left[\left(\frac{s+1}{c}, \frac{1}{c}\right); \frac{-a}{b^{1/c}}\right]. \quad (36)$$

The numerical values of these special functions can be computed using statistical softwares like MAPLE or MATHEMATICA for particular values of its parameters.
Using equations (25), (28), (31), (33) or (36) in (23) depending on the values of α and β gives the r^{th} raw moment of the ALIWD(ρ, σ, α, β) for various values of its parameters.

Incomplete Moments:
For r¿=i and y¿1, the rth incomplete moment $\Delta_r(y)$ can be obtained ase obtain an expression for the r^{th} incomplete moment of the ALIWD(ρ, σ, α, β) random variable U through the definition,

$$\Delta_r(y) = E(U^r/U \leq y) = \int_1^{y} u^r f_U(u))du \quad (37)$$
$$= \sum_{j=0}^{\infty} \frac{r^j}{j!} \int_0^{[\ln(y)]^{-1}} t^{-j}\left(\alpha\rho t^{\alpha-1} + \beta\sigma t^{\beta-1}\right) exp\left\{-\left(\rho t^{\alpha} + \sigma t^{\beta}\right)\right\} dt, \quad (38)$$

using the substitution $t = [\ln(u)]^{-1}$ and expanding the exponential term $exp[rt^{-1}]$. Substituting $t^{\alpha} = z$ in (38), we have

$$\Delta_r(y) = \sum_{j=0}^{\infty} \frac{r^j}{\alpha\, j!} \int_1^{\ln(y)^{-\alpha}} z^{(1-j)\alpha^{-1}-1}\left(\alpha\rho z^{(\alpha-1)\alpha^{-1}} + \beta\sigma z^{(\beta-1)\alpha^{-1}}\right)$$
$$\times exp\left\{-\left(\rho z + \sigma z^{\beta\alpha^{-1}}\right)\right\} dz.$$

On expanding the term $exp[-\rho z]$, we obtain

$$\Delta_r(y) = \sum_{j,k=0}^{\infty} \frac{r^j(-1)^k\rho^k}{j!k!} \left[\rho I_{(\ln(y)^{-\alpha},k-j\alpha^{-1},\sigma,\beta\alpha^{-1})} + \sigma\beta\alpha^{-1} I_{(\ln(y)^{-\alpha},(\beta-j)\alpha^{-1}+j-1,\sigma,\beta\alpha^{-1})}\right], \tag{39}$$

in which $I_{(y;\ s,\ \sigma,\ \beta\alpha^{-1})} = \int_0^y u^s exp\left[-\sigma u^{\beta\alpha^{-1}}\right]du$. When $\alpha \geq 1, \beta \geq 1$ are natural co-prime numbers, we have

$$I_{(y;\ s,\ \sigma,\ \beta\alpha^{-1})} = \int_0^y u^s G_{0,1}^{1,0}\left(\sigma u^{\beta\alpha^{-1}} \,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right) du, \tag{40}$$

for appropriate values of the parameters in the light of (30). Making use of Eq. (2.24.2.2) in Prudnikov et al.(1986, p 348), $I_{(y;s,\sigma,\beta\alpha^{-1})}$ can be expressed as

$$I_{(y;s,\sigma,\beta\alpha^{-1})} = \frac{\alpha y^{\beta(s+1)}}{\beta(2\pi)^{((\alpha-1)/2)}} G_{\beta,\beta+\alpha}^{\alpha,\beta}\left(\frac{\sigma^\alpha y^\beta}{\alpha^\alpha} \,\middle|\, \begin{matrix} \frac{-s}{\beta}, \frac{1-s}{\beta}, \ldots, \frac{\beta-s-1}{\beta}, - \\ 0, \frac{-s-1}{\beta}, \frac{s}{\beta}, \ldots, \frac{\beta-s-2}{\beta} \end{matrix}\right). \tag{41}$$

By using (40) and (41) in (39), we obtain the expression for the r^{th} incomplete moments of the ALIWD(ρ, σ, α, β) for various values of its parameters.

Remark 4. Using $\Delta_r(t)$, the mean deviations about the mean μ_1 and the median M of the ALIWD(ρ, σ, α, β) can be obtained as,

$$E(|U-\mu_1|) = 2\mu_1 F(\mu_1) - 2\Delta_1(\mu_1) \tag{42}$$

and

$$E(|U-M|) = \mu_1 - 2\Delta_1(M) \tag{43}$$

respectively, where $\Delta_1(.)$ is as given in (2), for $r = 1$.

3. Some structural properties

In this section we present certain results highlighting some structural properties of the ALIWD. The proofs of Theorems 7, 8 and 9 are omitted as they can be obtained directly by using the method of transformation of variables.

Theorem 3.1. *A random variable U has the ALIWD(ρ, σ, α, β) if and only if $Y_1 = [\ln(U)]^{-1}$ has the AWD(ρ, σ, α, β).*

Theorem 3.2. *A random variable U has the ALIWD(ρ, σ, α, β) if and only if $Y_2 = \ln[U]$ follows the Inverse Weibull Multiplicative Model (IWMM) of [11] having c.d.f.*

$$Q_5(y) = \exp\left\{-\left[\rho y^{-a} + \sigma y^{-b}\right]\right\}, \tag{44}$$

in which $a = \rho^{\alpha^{-1}}$ *and* $b = \sigma^{\beta^{-1}}$.

Theorem 3.3. *A random variable U has the ALIWD(ρ, σ, α, β) if and only if* $Y_3 = U^c$ *has the ALIWD*$(\rho^*, \sigma^*, \alpha, \beta)$ *with* $\rho^* = \rho c^{\alpha}$, $\sigma^* = \sigma c^{\beta}$, *for* $c > 0$.

Theorem 3.4. *For extremely small values of the random variable U, the ALIWD(ρ, σ, α, β) tends to the IWMM having c.d.f.* (44) *with* $a = \rho^{\alpha^{-1}}$ *and* $b = \sigma^{\beta^{-1}}$.

Proof: When $u = 1 + t$, for extremely small values of $t > 0$, the c.d.f. $F_U(u)$ of the ALIWD$(\rho, \sigma, \alpha, \beta)$ given in (5) can be written as

$$F_U(u) = \exp\left\{-\left(\rho[\ln(1+t)]^{-\alpha} + \sigma[\ln(1+t)]^{-\beta}\right)\right\}. \tag{45}$$

On expanding $\ln(1+t)$ in (45) and discarding terms with higher powers of t, we obtain the following representation of $F(t)$, which is the c.d.f. of the IWMM as given in (44).

$$F(t) = \exp\left[-\left(\rho t^{-\alpha} + \sigma t^{-\beta}\right)\right]$$

■

Theorem 3.5. *If U be any continuous random variable with c.d.f.* $F(u)$, *for every* $u \in (1, \infty)$, *then* $E\{h(U)/U \le y\} = h(y) + d$, *for* $h(U) = \left\{\rho[\ln(U)]^{-\alpha} + \sigma[\ln(U)]^{-\beta}\right\}$ *and* $d = 1$ *if and only if U has the ALIWD(ρ, σ, α, β).*

Proof: The proof follows from Theorem 9 of [22] (pp. 264) since $E(h(U)) = 1$, $\lim_{u\downarrow 1} h(u) = \infty$ and $\lim_{u\to\infty} h(u) = 0$, so that

$$\begin{aligned} F(u) &= \exp\left[\frac{1}{d}\,[h(\infty) - h(u)]\right] \\ &= \exp\left[-\left(\rho[\ln(u)]^{-\alpha} + \sigma[\ln(u)]^{-\beta}\right)\right], \end{aligned} \tag{46}$$

for $u \in (1, \infty)$, which is the c.d.f. of the ALIWD$(\rho, \sigma, \alpha, \beta)$. ■

4. Order statistics: Distribution and Moments

Let $U_{i:n}$ be the i^{th} order statistic based on a random sample $U_1, U_2 \ldots, U_n$ of size n from the ALIWD$(\rho, \sigma, \alpha, \beta)$ having p.d.f. $f_U(u) = f_U(u; \underline{\theta})$ for $\underline{\theta} = (\rho, \sigma, \alpha, \beta)$, as given in (6), c.d.f. $F_U(u)$ as given in (5) and r^{th} raw moment $\mu_r^{'} = \mu_r(\underline{\theta})$. This section provides expressions for the distribution and moments of $U_{i:n}$ through the following theorems when $U > 1$.

Theorem 4.1. *The p.d.f. of* $U_{i:n}$ *is given by*

$$f_{i:n}(u) = \sum_{j=0}^{i-1} \nu_{n:i:j}\, f_U(u; \underline{\theta}^*), \tag{47}$$

in which $\nu_{n:i:j} = \dfrac{i\binom{n}{i}\binom{n-i}{j}(-1)^{n-i-j}}{(n-j)}$, *and* $\underline{\theta}^* = ((n-j)\rho, (n-j)\sigma, \alpha, \beta)$.

Proof: By definition, the p.d.f. of $U_{i:n}$ can be written as

$$\begin{aligned} f_{i:n}(u) &= \frac{n!}{(i-1)!\,(n-i)!}\,[F_U(u)]^{i-1}\,[1-F_U(u)]^{n-i} f_U(u), \\ &= \sum_{j=0}^{i-1} \frac{n!}{(i-1)!\,(n-i)!} \binom{n-i}{j} (-1)^{n-i-j} \\ & \exp\left\{-(n-j-1)\left(\rho[\ln(u)]^{\alpha} + \sigma[\ln(u)]^{\beta}\right)\right\} f_U(u), \end{aligned} \tag{48}$$

by using the binomial theorem. Now (48) gives (47) in the light of (6). ■
The following corollaries and Theorem (4.5) follow directly from Theorem (4.1).

Corollary 4.2. *The p.d.f. of the smallest order statistic $U_{1:n} = \min(U_1, U_2, \ldots, U_n)$ is obtained as*

$$f_{1:n}(u) = \sum_{j=0}^{n-1} \binom{n}{j} (-1)^j f_U(u; \underline{\theta}_1^*), \tag{49}$$

in which $\underline{\theta}_1^* = ((j+1)\rho, (j+1)\sigma, \alpha, \beta)$, where $f_U(.)$ is as given in (6) *and $u > 1$,.*

Corollary 4.3. *For $u > 1$, the p.d.f. of the largest order statistic $U_{n:n} = \max(U_1, U_2, \ldots, U_n)$ is*

$$f_{n:n}(u) = f_U(u; \underline{\theta}_2^*), \tag{50}$$

in which $\underline{\theta}_2^* = (n\rho, n\sigma, \alpha, \beta)$, where $f_U(.)$ is as given in (6).

Corollary 4.4. *For $u > 1$, the p.d.f. of the median $U_{m+1:n}$, with $n = 2m+1$, is the following for $f_U(.)$ as given in* (6).

$$f_{(m+1:n)}(u) = \sum_{j=0}^{m} \frac{(-1)^j (2m+1)\binom{2m}{m}\binom{m}{j}}{m+k+1} f_U(u; \underline{\theta}_3^*), \tag{51}$$

in which $f_U(u; \underline{\theta}_3^*) = f_U((m+j-1)\rho, (m+j-1)\sigma, \alpha, \beta)$.

Theorem 4.5. *The r^{th} raw moment of $U_{i:n}$ is given by*

$$\mu_{r(i:n)}(u) = \sum_{j=0}^{i-1} \nu_{n:i:j}\, \mu_r(\underline{\theta}^*), \tag{52}$$

where $\nu_{n:i:j}$ and $\underline{\theta}^*$ are as defined in (47).

5. Maximum Likelihood Estimation

The ML estimation of the parameters of the ALIWD(ρ, σ, α, β) is discussed in this section. Let $U_1, U_2, \ldots, U_n$ be a random sample taken from a population following the ALIWD(ρ, σ, α, β). For the vector of parameters $\underline{\theta} = (\rho, \sigma, \alpha, \beta)$, the log-likelihood function is given by

$$\ell(\underline{\theta}) = \sum_{i=1}^{n} \Big\{ \ln(u_i^{-1}) + \ln\Big(\alpha\rho[\ln(u_i)]^{-\alpha-1} + \beta\sigma[\ln(u_i)]^{-\beta-1}\Big) - \Big(\rho[\ln(u_i)]^{-\alpha} + \sigma[\ln(u_i)]^{-\beta}\Big)\Big\}. \tag{53}$$

On differentiating the log-likelihood function (53) with respect to the parameters ρ, σ, α and β respectively and equating to zero, we obtain the following likelihood equations in which $\varphi_i = [\ln(u_i)]^{-\alpha}$, $v_i = [\ln(u_i)]^{-\beta}$, $w_i = \alpha\rho\varphi_i + \beta\sigma v_i$ and $u_i^* = \ln(u_i)$.

$$\sum_{i=1}^{n} \left[\varphi_i\left(\frac{\alpha}{w_i} - 1\right)\right] = 0, \tag{54}$$

$$\sum_{i=1}^{n} \left[v_i\left(\frac{\beta}{w_i} - 1\right)\right] = 0, \tag{55}$$

$$\sum_{i=1}^{n} \left\{\rho\,\varphi_i\left[\frac{1-\alpha\ln(u_i^*)}{w_i} + \ln(u_i^*)\right]\right\} = 0 \tag{56}$$

and

$$\sum_{i=1}^{n} \left\{\sigma\,v_i\left[\frac{1-\beta\ln(u_i^*)}{w_i} + \ln(u_i^*)\right]\right\} = 0. \tag{57}$$

The observed Fisher information matrix is derived as

$$I_{\underline{\theta}} = ((I_{ij})), \tag{58}$$

where the elements of $I_{\underline{\theta}}$ are as given below, for i, j= $1, 2, \ldots, n$.

$$I_{11} = \frac{d^2\ell(\underline{\theta})}{d\rho^2} = -\sum_{i=1}^{n} \frac{\alpha^2\varphi_i^2}{w_i^2}, \tag{59}$$

$$I_{12} = \frac{d^2\ell(\underline{\theta})}{d\rho d\sigma} = -\sum_{i=1}^{n} \frac{\alpha\beta\varphi_i v_i}{w_i^2}, \tag{60}$$

$$I_{13} = \frac{d^2\ell(\underline{\theta})}{d\rho d\alpha} = \sum_{i=1}^{n} \varphi_i \left\{\frac{\left[1-\alpha\ln(u_i^*)\right]}{w_i}\left[1 - \frac{\rho\alpha\varphi_i}{w_i}\right] - \ln(u_i^*)\right\}, \tag{61}$$

$$I_{14} = \frac{d^2\ell(\underline{\theta})}{d\rho d\beta} = -\sum_{i=1}^{n} \frac{\alpha\sigma\varphi_i v_i\left[1-\beta\ln(u_i^*)\right]}{w_i^2}, \tag{62}$$

$$I_{22} = \frac{d^2\ell(\underline{\theta})}{d\sigma^2} = -\sum_{i=1}^{n} \frac{\beta^2 v_i^2}{w_i^2}, \tag{63}$$

$$I_{23} = \frac{d^2\ell(\underline{\theta})}{d\sigma d\alpha} = -\sum_{i=1}^{n} \frac{\beta\rho\varphi_i v_i \left(1-\alpha\ln(u_i^*)\right)}{w_i^2}, \tag{64}$$

$$I_{24} = \frac{d^2\ell(\underline{\theta})}{d\sigma d\beta} = \sum_{i=1}^{n} v_i \left\{ \frac{\left(1-\beta\ln(u_i^*)\right)}{w_i}\left[1-\frac{\sigma\beta v_i}{w_i}\right] - \ln(u_i^*) \right\}, \tag{65}$$

$$I_{33} = \frac{d^2\ell(\underline{\theta})}{d\alpha^2} = \sum_{i=1}^{n} \rho y_i \left\{ \frac{\ln(u_i^*)\left[\alpha\ln(u_i^*)-2\right]}{w_i} + \frac{\rho\varphi_i\left[1-\alpha\ln(u_i^*)\right]^2}{w_i^2} - [\ln(u_i^*)]^2 \right\}, \tag{66}$$

$$I_{34} = \frac{d^2\ell(\underline{\theta})}{d\alpha d\beta} = -\sum_{i=1}^{n} \frac{\rho\sigma\varphi_i v_i[1-\alpha\ln(u_i^*)][1-\beta\ln(u_i^*)]}{w_i^2} \tag{67}$$

and

$$I_{44} = \frac{d^2\ell(\underline{\theta})}{d\beta^2} = \sum_{i=1}^{n} \sigma v_i \left\{ \frac{\ln(u_i^*)\left[\beta\ln(u_i)-2\right]}{w_i} + \frac{\sigma v_i\left[1-\beta\ln(u_i^*)\right]^2}{w_i^2} - [\ln(u_i^*)]^2 \right\}. \tag{68}$$

6. Applications

In this section the utility of the ALIWD(ρ, σ, α, β) is demonstrated with the help of the following two datasets.
Data Set 1: The data pertains to the survival of 40 patients suffering from Leukemia, obtained from the Ministry of Health Hospitals in Saudi Arabia ([1]).
115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852.
Data Set 2: Data on remission times for a group of leukaemia patients given the drug 6-MP from [18].
6, 7, 10, 13, 16, 22, 23
The values of the K-S statistics corresponding to the ALIWD(ρ, σ, α, β) for the two datasets along with the corresponding critical values at 1% level are provided in Table (1).

Table 1. K-S statistics for the ALIWD(ρ, σ, α, β) corresponding to Data Set 1 and 2.

K-S Statistic	Data Set 1	Data Set 2
Calculated Value	0.2325	0.1569
Critical Value	0.2521	0.5758

We have obtained the ML estimates of the parameters of the ALIWD(ρ, σ, α, β) by using **R software** in the case of the above two datasets and the same is presented in Table (2) along with the values of the corresponding standard errors and p-values.

Table 2. Fitted values of the parameters of ALIWD(ρ, σ, α, β) corresponding to Data Set 1 and Data Set 2.

Data Set 1

Parameter	Estimate	Std Error	t-Value	p-value
ρ	51755	0.28×10^{-08}	Inf	$< 2.2 \times 10^{-16}$
σ	51735	0.62×10^{-05}	Inf	$< 2.2 \times 10^{-16}$
α	6.1706	0.4536	13.604	$< 2.2 \times 10^{-16}$
β	6.1710	0.4532	13.616	$< 2.2 \times 10^{-16}$

Data Set 2

Parameter	Estimate	Std Error	t-Value	p-value
α	5.1335	0.6924	7.4134	$< 2.2 \times 10^{-16}$
β	14.3584	2.1×10^{-06}	inf	$< 2.2 \times 10^{-16}$
ρ	60.2710	23.1330	2.6054	0.00917
σ	0.5284	8.4×10^{-07}	Inf	$< 2.2 \times 10^{-16}$

The variance-covariance matrices corresponding to Data Set 1 and 2 respectively are

$$\Sigma_1 = \begin{bmatrix} 0.20575 & & & \\ -0.18975 & 0.20539 & & \\ 0.00000 & 0.00000 & 0.0000 & \\ 0.00000 & 0.00000 & -17.59219 & 0.00000 \end{bmatrix} \tag{69}$$

and

$$\Sigma_2 = \begin{bmatrix} 0.47951 & & & \\ 0.0000 & 0.0000 & & \\ 11.02883 & 0.0000 & 535.13812 & \\ 0.0000 & 281.4750 & 0.0000 & 0.0000 \end{bmatrix} \tag{70}$$

For comparison, we have fitted the following distributions along with the ALIWD: the log generalized inverse Weibull distribution (LGIWD) of [5], the IWD, the Weibull distribution (WD), the LIWD, the reduced log generalised inverse Weibull distribution (RLGIWD) of [5] as well as a left truncated version of the IWD truncated at 1, called as the 'left truncated inverse Weibull distribution (LTIWD)'. The distributions are compared using certain information criteria like 'the Akaike information criteria (AIC)', 'the Bayesian information criteria (BIC)', 'the corrected Akaike information criteria (AICc)' and 'the consistent Akaike information criteria (CAIC)' and the numerical results obtained for the Datasets 1 and 2 are summarised in Table (3) and Table (4) respectively. We have obtained cumulative probability plots in Figures (5) and (7) and WPPs corresponding to the fitted models are presented in Figures (6) and (8) respectively for graphical comparison. From these tables and figures, it can be observed that the ALIWD(ρ, σ, α, β) gives relatively better fit to the Datasets 1 and 2 as compared to the other models.

Table 3.: Fitting various distributions to Data Set 1

Model	Estimates	Log-Likelihood	AIC	BIC	AICc	CAIC
ALIWD	ρ = 51755 σ = 51735 α = 6.1706 β = 6.191	-322.794	653.589	660.344	654.732	664.344
LGIWD	μ = 2300.5860 σ = 2877.8820	-350.173 -394.603	706.3460 791.207	711.4120 792.3502	707.3460 791.962	714.412 793.962

Continued $\cdots$

Model	Estimates	Log-Likelihood	AIC	BIC	AICc	CAIC
	$\gamma = 0.6558$					
IWD	$c = 0.1960$					
WD	$\alpha = 0.116$	-407.99	823.981	825.124	830.736	834.736
LIWD	$c = 0.701$	-419.868	841.736	843.424	841.841	844.424
LTIWD	$c = 0.196$	-376.710	755.430	757.470	755.500	758.470
RLGIWD	$\gamma = 3200.597$	-43587.230	87176.460	87176.565	87178.150	87179.150

Table 4.: Fitting various distributions to Data Set 2

Model	Estimates	Log-Likelihood	AIC	BIC	AICc	CAIC
ALIWD	$\rho = 60.2710$	-23.180	54.360	54.150	74.360	58.150
	$\sigma = 0.5280$					
	$\alpha = 5.1330$					
	$\beta = 14.3580$					
LTIWD	$c = 0.53448$	-30.077	62.154	62.100	62.954	63.100
LIWD	c = 1.4871	-32.451	66.902	66.848	67.702	67.848
IWD	$\alpha = 0.53448$	-33.287	68.575	68.521	69.375	69.521
WD	$\alpha = 0.31126$	-35.784	73.569	73.515	74.369	74.515
RLGIWD	$\gamma = 2018.90$	-50.669	103.339	103.285	104.139	104.285
LGIWD	$\mu = 1161.244$	-56.400	118.801	118.639	126.801	121.639
	$\sigma = 0.8649$					
	$\gamma = 182.0800$					

7. Simulation

In order to examine the asymptotic behaviour of the MLEs of the parameters of the ALIWD(ρ, σ, α, β), we carry out a simulation study by generating AWD(ρ, σ, α, β) observations (Y) with the help of MATHEMATICA and transforming them to the corresponding ALIWD(ρ, σ, α, β) observations (U) using the transformation $U = \exp(Y^{-1})$. Observations were generated for the following two sets of parameters (1) $\rho = 1, \sigma = 1, \alpha = 0.001, \beta = 0.1$ and (2) $\rho = 1.5, \sigma = 1.2, \alpha = 8, \beta = 0.8$ corresponding to the upside-down bathtub shape and decreasing hazard rate shape as seen in Figure (2). According to [7], a maximum of 200 bootstrap samples are required to obtain a good estimate of the variance of an estimator. Hence we have considered 200 bootstrap samples of sizes 50, 500 and 1000 for comparing the performances of the different MLEs mainly with respect to their mean values and MSEs. The average bias of the estimates and average MSEs over 200 replications are calculated for different cases and presented in Table (5). From Table (5), it can be observed that as sample size increases the bias decreases while the MSEs of the estimators are in decreasing order.

Figure 5. Cumulative probability plots of various distributions corresponding to the Data Set 1.

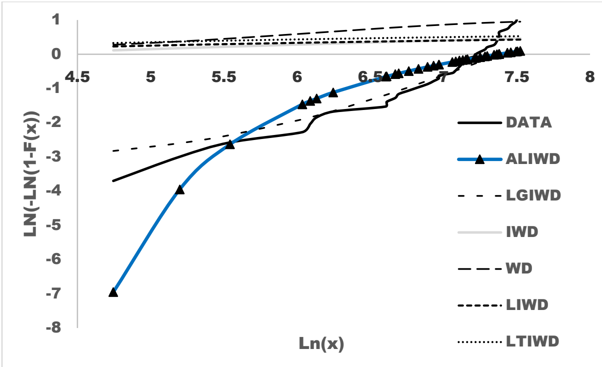
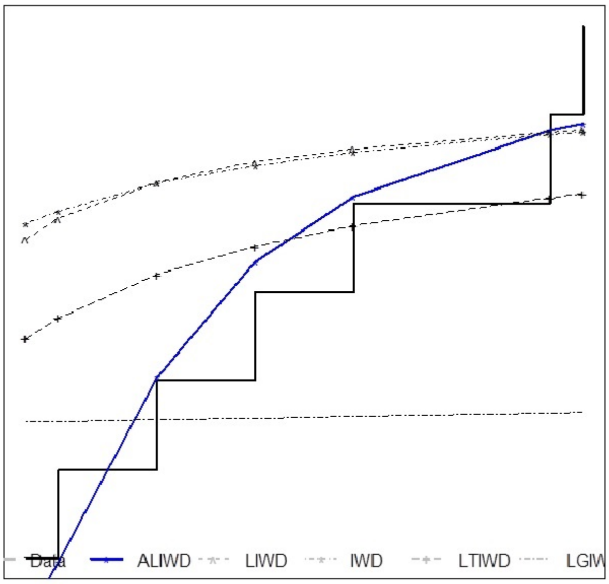


Figure 6. Weibull Probability Plots of various distributions corresponding to Data Set 1.



(a)

Figure 7. Cumulative probability plots of various distributions corresponding to Data Set 2.

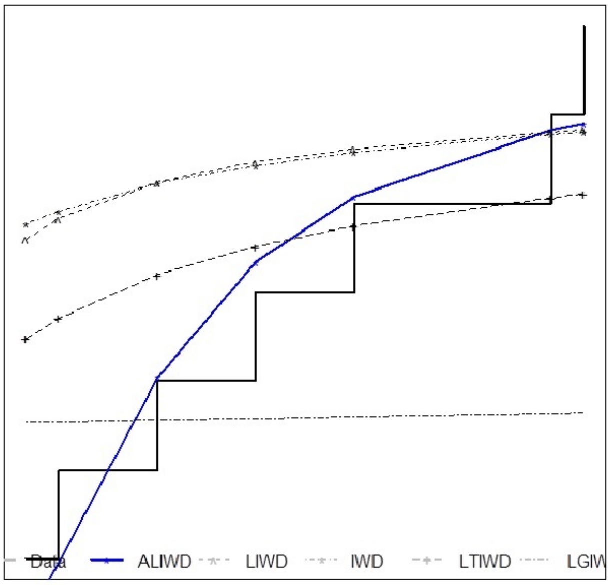
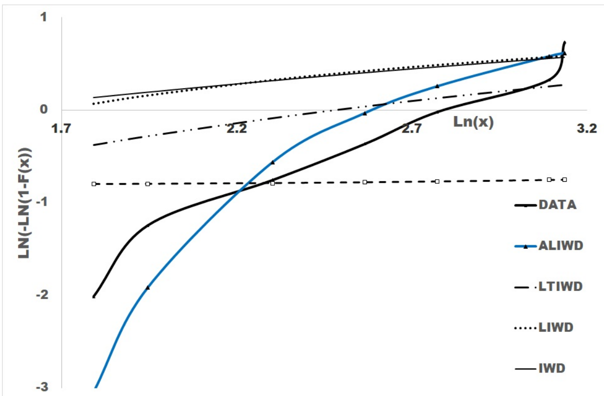


Figure 8. Weibull Probability Plots of various distributions corresponding to Data Set 2.



(a)

Table 5. Average bias and mean squared errors (within parenthesis) of the MLEs of the parameters of the ALIWD(ρ, σ, α, β) based on simulated datasets corresponding to parameter sets (1) $\rho = 0.25, \sigma = 1.25, \alpha = 0.001, \beta = 0.1$ and (2) $\rho = 1.5, \sigma = 1.2, \alpha = 8, \beta = 0.8$.

Parameter set	Sample Size	ρ	σ	α	β
Set 1	50	0.5315	0.5428	0.3312	0.3415
		(0.00015)	(0.00023)	(0.05648)	(0.05692)
	500	0.2892	0.2699	0.1824	0.1588
		(0.00011)	(0.00015)	(0.02158)	(0.03114)
	1000	0.000112	0.000871	0.00001	- 0.00001
		(0.000001)	(0.000001)	(0.000011)	(0.000045)
Set 2	50	0.0288	0.02198	0.3928	0.3997
		(0.005490)	(0.006884)	(0.002588)	(0.005124)
	500	0.00115	0.000968	0.12587	0.11281
		(0.00012)	(0.00124)	(0.00091)	(0.00044)
	1000	0.00012	-0.00001	0.00536	0.00052
		(0.000082)	(0.000012)	(0.000011)	(0.000005)

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